1. Introduction to Problem

- Data assimilation is the process of optimally combining observations and model predictions to produce a best estimate of a system.
- Mathematically, the problem of finding the best state estimate, \( x_k \), can be stated as follows:
  \[
  \min J(x) = \frac{1}{2} \lVert z - Hx \rVert^2_{R^{-1}} + \frac{1}{2} \lVert y - x \rVert^2_{Q^{-1}}.
  \]
Subject to \( x_k = y_k + \nu_k \) where the model prediction, \( y_k = Ax_{k-1} \), is subject to model error \( \nu_k, E[\nu] = 0 \), and observations in time, \( z_k \), are related to the state, \( Hx \), and contain measurement error \( E[e] = 0 \).
- Common solution methods include the Kalman Filter and 4D-Var (e.g. see [1]).
- The SMART Filter was developed as an inverse method for dynamic medical image reconstruction [2].
- [2] presents the SMART Filter as a computationally efficient alternative to the Kalman Filter.
- Here we seek to apply the SMART Filter to a linear advection data assimilation problem.

2. SMART Filter Derivation

- The SMART algorithm [3] solves the regularization problem
  \[
  \min_{x \geq 0} F(x) = \alpha KL(P, x, d) + (1 - \alpha) KL(x, y)
  \]
for a linear system \( d = Px \), and uses cross entropy (or Kullback-Leibler) distance
  \[ KL(a, b) = a \log \frac{a}{b} - a + b, \quad a, b \geq 0, \quad \text{and, parameter } 0 \leq \alpha \leq 1 \]
- [2] develops this into a filtering algorithm to solve a temporal regularization problem
  \[
  \min_{x \geq 0} F(x_k) = KL(H_k x_k, z_k) + KL(x_k, y_k)
  \]
- This function can be weighted according to model error covariance, \( Q \), and observation error covariance, \( R \):
  \[
  F(x) = KL_R^{-1}(Hx, z) + KL_Q^{-1}(x, y)
  \]
- A Cholesky decomposition, where \( U_1 \) and \( U_2 \) are non-negative diagonal matrices,
  \[
  R^{-1} = U_1^T U_1, \quad Q^{-1} = U_2^T U_2
  \]
allows us to rewrite the cost function as:
  \[
  F(x) = KL(U_1 H x, U_1 z) + KL(U_2 x, U_2 y)
  \]
- For our example we take \( Q \) and \( R \) to be of the form
  \[
  R_k = \sigma^2_k I, \quad U_1 = \frac{1}{\sigma_k} I
  \]
  \[
  Q_k = \sigma^2_k I, \quad U_2 = \frac{1}{\sigma_k} I
  \]
- Hence
  \[
  F(x) = KL_R^{-1}(H x, z) + KL_Q^{-1}(x, y)
  \]
  \[
  = KL \left( \frac{1}{\sigma} H x, \frac{1}{\sigma} z \right) + KL \left( \frac{1}{\sigma} \frac{1}{\sigma} y \right)
  \]
Which can easily be manipulated into the required form:
  \[
  F(x) = \frac{1}{\sigma} \left( \frac{\sigma - 1}{\sigma} KL \left( \frac{\sigma}{\sigma - 1} H x, \frac{\sigma}{\sigma - 1} z \right) + \frac{1}{\sigma} KL(x, y) \right)
  \]

3. Experiment and Results

- We illustrate SMART using a data assimilation problem given in [4], whereby a smooth periodic pseudorandom wave is advected on a periodic domain of length 200 m.
- The model has a constant advection speed of 1 m/s, the grid spacing is equal to 1 m, and a time step of 1 s is used.
- The initial condition is offset from the truth by adding a normally distributed error with 0 mean, variance 1, and a de-correlation length of 20 m.
- We take 20 equally spaced observations of the truth every 6 seconds.
- In both experiments we applied normally distributed observation error of mean 0 and variance 0.01.
- The solution of the model is known, thus we are able to run simulations without model error (graphs on left panel) and with model error (graphs on right panel).
- The model error is normally distributed of mean 0, variance 0.001, and de-correlation length 20 m.
- The length of the total integration is 300 seconds.

4. Future Steps

- Currently the SMART filter as implemented here assumes that the model error applied at each update
  \[
  x_k = A_k x_{k-1} + \nu_k
  \]
is white noise therefore to improve the effectiveness of the SMART filter in our example we should first apply a pre-whitening process, i.e.
  \[
  v = \Lambda^{\frac{1}{2}} E^T \mu
  \]
where \( E \) is the orthonormal matrix of eigenvectors and \( \Lambda \) is the diagonal matrix of positive eigenvalues of the spectral decomposition of the definite positive covariance matrix \( Q \) (see [2] for details).
- We would also like to apply the SMART filter to a nonlinear system.

Sources


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