The SMART Filter as an Efficient Data Assimilation Method in Geophysical Fluid Dynamics

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1. Introduction

- Data assimilation (DA) is the process of optimally combining observations and model predictions to produce a best estimate of a system
- DA is used extensively in geophysical fluid dynamics, such as numerical weather prediction[1]

2. Statement of Problem

Given a discrete nonlinear system modelled by

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k) + \epsilon_k, \qquad k = 0, \dots, N-1$$

where x_k denotes the model state, u_k denotes the inputs to the system at time k, f_k is a function for the evolution of the model state from time k to k+1, and ε_k represents the model error and observations, y_k, related to the system states by

$$\mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k) + \delta_k, \qquad k = 0, \dots, N-1$$

where δ_k are measurement errors, then we seek to minimize, with respect to x_0 the following cost function:

$$\mathcal{J} = \frac{1}{2}(\mathrm{x}_0 - \mathrm{x}_0^b)^T B_0^{-1}(\mathrm{x}_0 - \mathrm{x}_0^b) + \\ + \frac{1}{2}\sum_{k=0}^{N-1}(\mathrm{h}_k(\mathrm{x}_k) - \mathrm{y}_k)^T R_k^{-1}(\mathrm{h}_k(\mathrm{x}_k) - \mathrm{y}_k) + \\ + \frac{1}{2}\sum_{k=0}^{N-1}\epsilon_k^T Q_k^{-1}\epsilon_k,$$

Where X_0^b is the background initial guess, $B_0 = E[(x_0 - x_0^b)(x_0 - x_0^b)^T]$ = background error covariance matrix, $Q_k = E(\varepsilon_k \varepsilon_k^T)$ = model error covariance matrix, and $R_k = E(\delta_k \delta_k^T)$ = observation error covariance matrix.

3. Kalman Filter and Optimal Interpolation

The cost function can be minimized directly and solved sequentially, such as is done with the Kalman filter[2]:

$$x_a = x_0^b + \mathbf{K}(y_k - \mathbf{h}[x_0^b])$$

$$\mathbf{K} = \mathbf{P}_f(k)\mathbf{h}^T(k)[\mathbf{h}(k)\mathbf{P}_f(k)\mathbf{h}^T(k) + R(k)]^{-1}$$

$$\mathbf{P}_f = \mathbf{f}_{i\longrightarrow i+1}\mathbf{P}_a\mathbf{f}_{k\longrightarrow k+1}{}^T + \mathbf{Q}(k)$$

$$\mathbf{P}_a(k) = [\mathbf{I} - \mathbf{K}(k)\mathbf{h}(k)]\mathbf{P}_f(k)$$

Optimal Interpolation approximates the Kalman filter by keeping a constant P_f

4. SMART Filter Solution

- The Kalman filter is computationally expensive
- Matrix inversions and matrix-matrix multiplications make it intractable for very large systems
- The SMART filter was developed as an alternative to the Kalman filter for use in medical imaging and is computationally faster[3]
- It is derived by minimizing the following cost function: $F(x) = KL_{R^{-1}}(hx, y) + KL_{Q^{-1}}(x, z)$

where $z = Mx_k$, and where

$$KL(a,b) = \sum_{j=1}^{N} KL(a_j, b_j) = \sum_{j=1}^{N}\left(a_j \log\frac{a_j}{b_j} + b_j - a_j\right)$$

is the Kullback-Leibler distance between two non-negative vectors. The SMART Filter is outlined as follows, with the analysis being x^∞ :

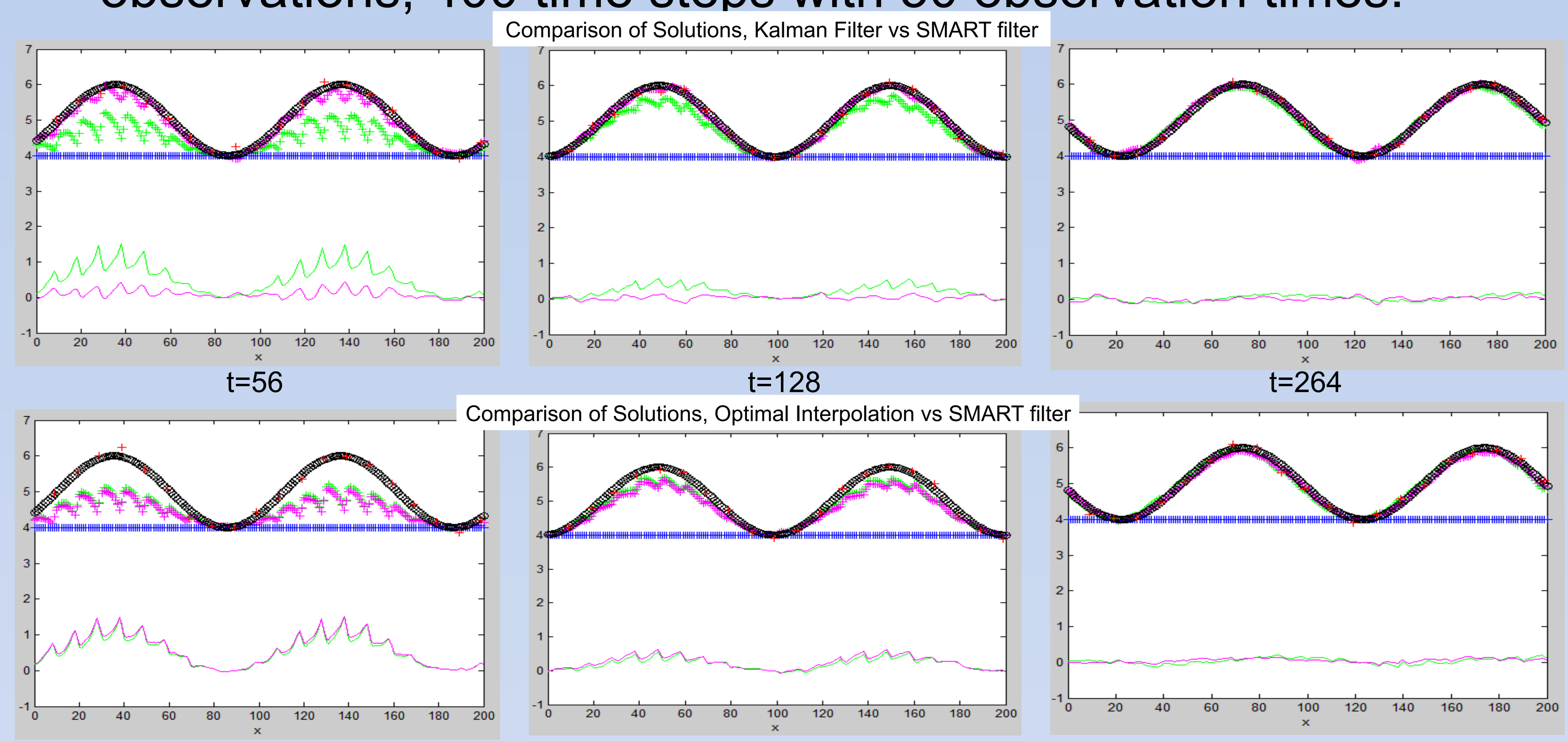
$$\alpha = \frac{\sigma_k - 1}{\sigma_k} \quad d = R_k^{-1/2} z_k \quad P = R_k^{-1/2} H_k$$

$$r_{ij}^{l+1} = P_{ij}\log\frac{d_i}{(Px^l)_i}$$

$$x_j^{l+1} = (x_j^l)^\alpha((y_k)_j)^{1-\alpha} exp\left(\alpha\sum_{i=1}^{M} r_{ij}^{l+1}\right)$$

5. Results

- We applied the SMART filter to a 1D linear advection problem
- The simulated assimilation uses 200 domain points, 20 spatial observations, 400 time steps with 50 observation times.



Blue=initial guess (background), magenta = Kalman and Optimal Interpolation analysis, green = SMART analysis, black = true function, bottom lines = error between true function and the Kalman/ Optimal Interpolation and SMART analyses

- Computation times:
- 400 variable system: SMART filter-70 seconds, Kalman filter-242, Optimal Interpolation-214
- 200 variable system: SMART filter-0.34 seconds, Kalman filter-1.70, Optimal Interpolation-0.31

6. Conclusions

- The SMART filter's computation time is over 3 times faster than the Kalman filter
- The SMART filter's computation time was also around 2-3 times as fast as Optimal Interpolation with sufficiently large systems as well as converging to the true solution faster
- The SMART filter could be used as an alternative to the Kalman filter when computational time is a concern, or as an alternative to Optimal Interpolation for faster convergence to solution

References

[1] E. Kalnay, Atmospheric Modelling, Data Assimilation and Predictability. Cambridge University Press, 2003.

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