The Evolution of Radially Symmetric Ice Sheet using Level Set Method

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Introduction

- Ice is an incompressible and non-Newtonian viscous fluid with extremely low Reynolds number flow.
- A level set method handles topological changes of glaciers and the evolution of the ice-air or ice-water interface.
- A level set method for modeling glacier is developing for advancing and retreating glaciers and initialization of ice-sheet models.

Ice-sheet Model

Ice Flow Equations

$$\nabla\cdot\boldsymbol{v}=0$$
$$\rho(\boldsymbol{v}_t+\boldsymbol{v}\cdot\nabla\boldsymbol{v})=-\nabla p+\nabla\cdot(\nu\nabla\boldsymbol{v})+\rho\boldsymbol{g}$$

Stokes Approximation

Ice is a slow moving fluid.

$$\rho(\boldsymbol{v}_t+\boldsymbol{v}\cdot\nabla\boldsymbol{v})\approx 0.$$

$$\begin{aligned}\text{(incompressibility)}\quad & \nabla\cdot\boldsymbol{v}=0\\ \text{(stress balance)}\quad & 0=-\nabla p+\nabla\cdot\tau_{ij}+\rho\boldsymbol{g}\\ \text{(Glen's flow law)}\quad & \mathcal{D}v_{ij}=A\tau^{n-1}\tau_{ij}\end{aligned}$$

- $\mathcal{D}v_{ij}$ strain rate tensor
- τ magnitude of τ_{ij}: $\tau^2=\frac{1}{2}\tau_{ij}\tau_{ji}$

Shallow Ice Approximation (SIA)

$$H_t=M+\nabla\cdot(D\nabla h)$$

- $\Gamma=\frac{2A}{n+2}(\rho g)^n$ and $D=\Gamma H^{n+2}|\nabla h|^{n-1}$
  - A, ice softness in Glen's flow law
  - $n=3$, in Glen's flow law
  - M, climate mass balance
  - ρ, density of ice (910 kg m^{-3})

Horizontal ice velocity,

$$u(r,z,t)=-\frac{2A(\rho g)^n}{n+1}\left[H^{n+1}-(h-z)^{n+1}\right]|h_r|^{n-1}h_r-\left(\frac{\rho g H h_r}{C}\right)^{1/m}$$

- C, bed friction parameter
- $m=\frac{1}{n}$, bed friction exponent

and vertical velocity,

$$w(r,z,t)=-\int_{b(r)}^{z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru(r,z',t)\right)\right)dz'$$

The profile of the steady state ice sheet is

$$h(r)=\left(\left(\frac{2(n+1)}{n\rho_i g}\right)^n\frac{n+2}{2A}\right)^{\frac{1}{2(n+1)}}\left(\int_r^R\left(\frac{1}{\xi}\int_0^{\eta}M(\eta)\eta\,d\eta\right)^{\frac{1}{n}}d\xi\right)^{\frac{n}{2(n+1)}}$$

where surface mass balance $M(r)$ is considered as independent of time and the bedrock is flat.

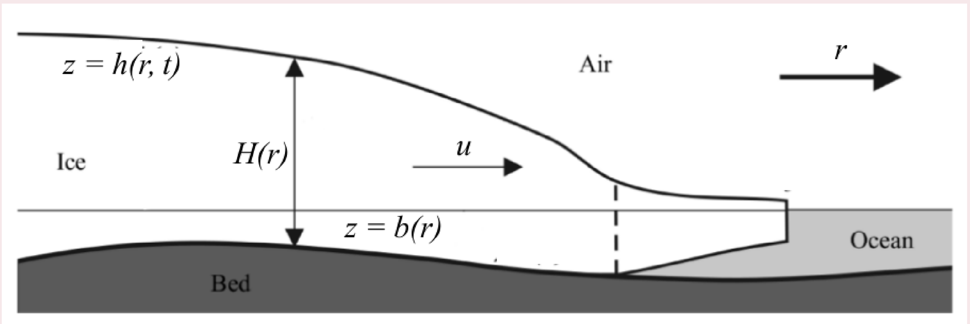


Figure 1: A marine ice sheet profile, image taken from Schoof [4].

Level Set Method

Level set function φ has the following properties:

$$\begin{aligned}\varphi(\boldsymbol{x},t)<0 \quad &\text{for}\quad \boldsymbol{x}\in\Omega_i\\ \varphi(\boldsymbol{x},t)>0 \quad &\text{for}\quad \boldsymbol{x}\in\Omega_c\\ \varphi(\boldsymbol{x},t)=0 \quad &\text{for}\quad \boldsymbol{x}\in\partial\Omega,\end{aligned}$$

where Ω_i represents the region inside ice and $\partial\Omega$ represents the ice-air interface. The level set function φ can define as a signed distance function,

$$\varphi(\boldsymbol{x},t)=\begin{cases}d(\boldsymbol{x},\partial\Omega), & \text{if}\quad \boldsymbol{x}\in\Omega_i\\ -d(\boldsymbol{x},\partial\Omega), & \text{if}\quad \boldsymbol{x}\in\Omega_c.\end{cases}$$

Now, the ice-air interface is expressed by,

$$\frac{\partial\varphi}{\partial t}+F\|\nabla\varphi\|=0,$$

where F is the speed in the normal direction and

$$F=(\boldsymbol{v}+b_{surf}(\boldsymbol{x},t)\hat{z})\cdot\frac{\nabla\varphi}{\|\nabla\varphi\|}.$$

- b_{surf} the surface mass-balance function in the vertical direction.

The ice velocity components and the ice thickness are only defined on Ω_i and need to be extended onto Ω_c.

$$\nabla F^{ext}\cdot\nabla\varphi=0.$$

Example: A test case of the Level set method [3]

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial r}-w=b_{surf}(r,t)$$

- $u(r,z)=r^2+z^2,\ w(r,z)=0$
- $h(r,t)=r-r^2+rt$
- $b_{surf}=r+(r^2+z^2)(1-2r+t)$
- Initial level set function $\varphi(r,z,0)=z-r+r^2$.

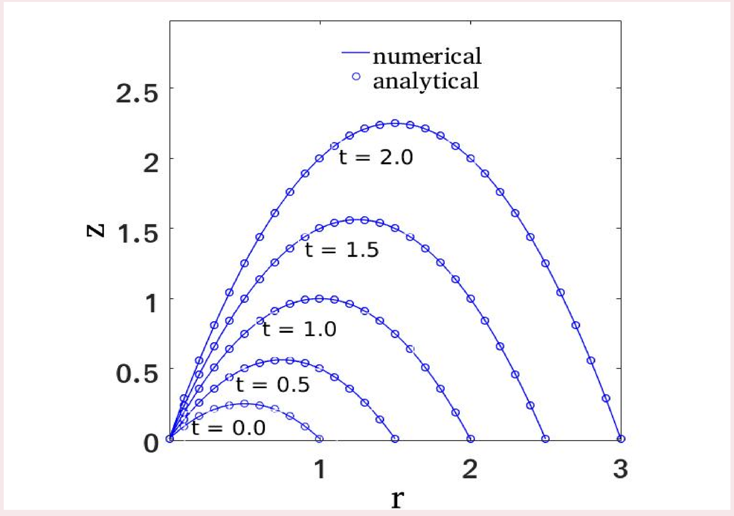
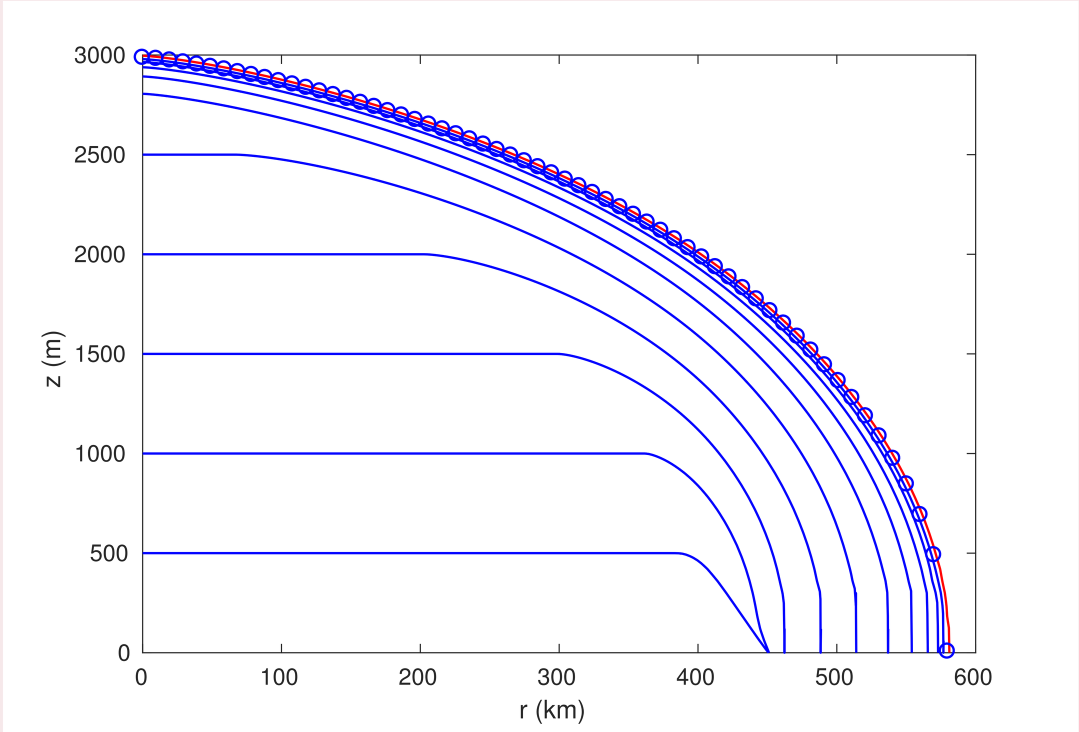


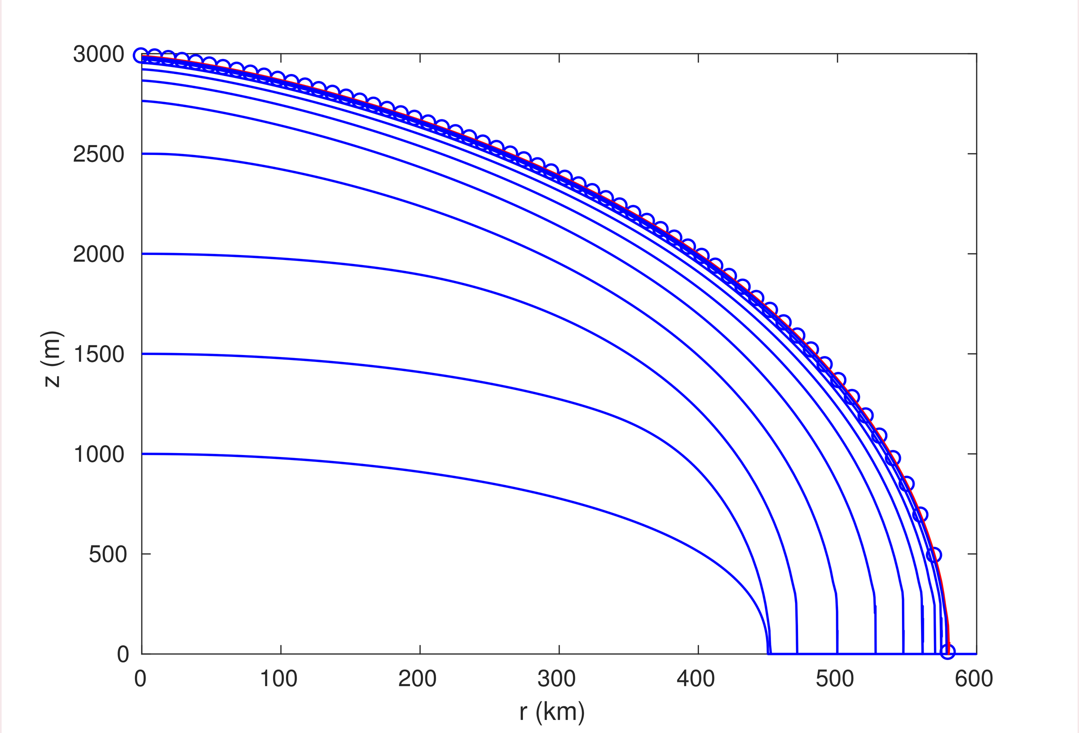
Figure 2: Evolution of the interface subject to an imposed velocity field and surface mass balance.

Results

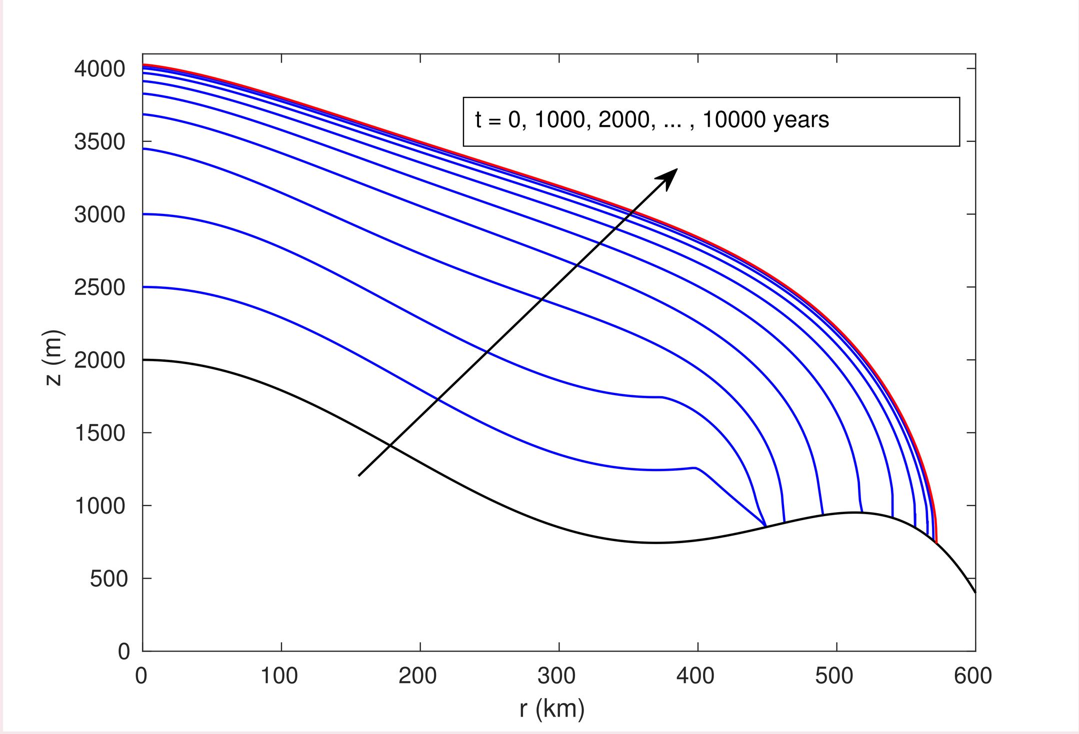
Experiment 1: flat bedrock; no-sliding; no initial ice mass



Experiment 2: flat bedrock; no-sliding; initialise ice mass;



Experiment 3: non-flat bedrock; no-sliding; no initial ice mass



Experiment 4: non-flat bedrock; sliding; no initial ice mass

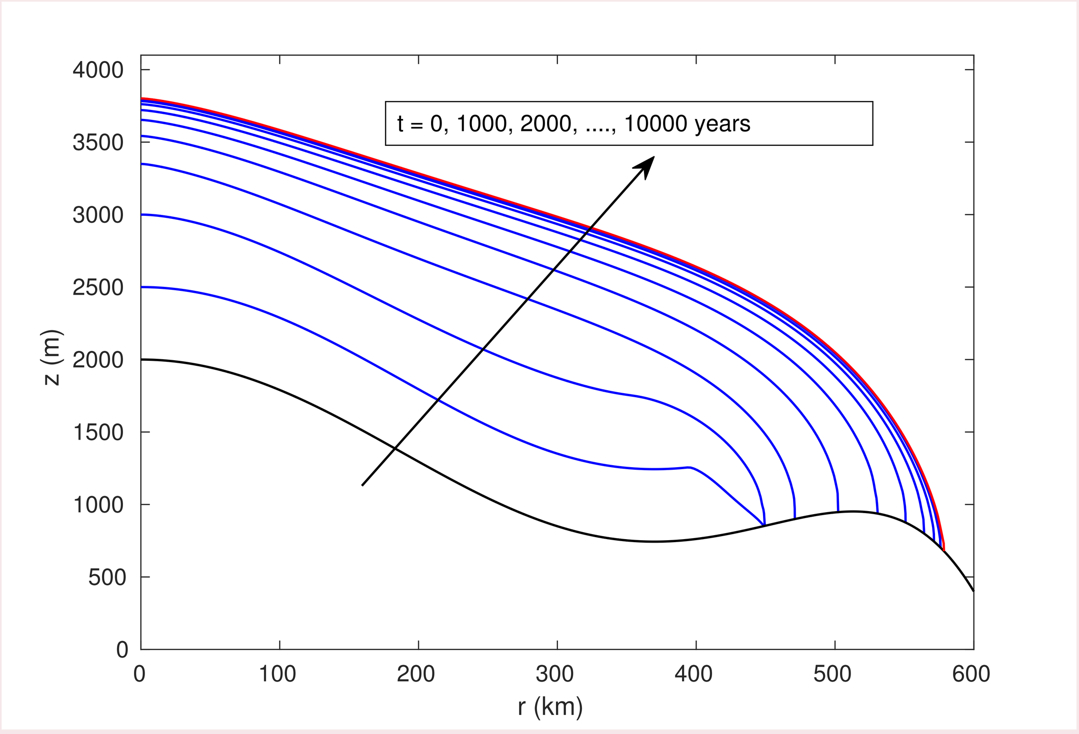


Figure 3: **Evolution of the ice sheet** o: exact; red: steady state; blue: interim state of every 1000 year period; black: bedrock

EISMINT intercomparison project

The EISMINT (European Ice Sheet Modeling INiTiative) intercomparison project used the following constant-in-time surface mass balance function:

$$M(r)=\min\left(0.5\text{ m yr}^{-1},10^{-2}\text{ m yr}^{-1}\text{km}^{-1}\cdot(450\text{ km}-r)\right)$$

Flat bedrock experiments:

- **Experiment 1:**
  - no initial ice mass; no-sliding
- **Experiment 2:**
  - no-sliding; initialise the following profile:

$$h_0(r)=1000\left(1-\left(\frac{r}{450\text{ km}}\right)\right)\text{ m}$$

Non-flat bedrock experiments: Consider the following fixed bedrock elevation [1]

$$b(r)=2000-2000\left(\frac{r}{300\text{ km}}\right)^2+1000\left(\frac{r}{300\text{ km}}\right)^4-150\left(\frac{r}{300\text{ km}}\right)^6\text{ m}$$

- **Experiment 3:**
  - no initial ice mass; no-sliding
- **Experiment 4:**
  - no initial ice mass; including sliding
  - bed friction parameter, $C=7.624\times10^6\text{ Pa m}^{-1/3}\text{ s}^{1/3}$

Summary

1. A level set method is implemented for the solution of an ice-air interface subject to an imposed velocity field and surface mass balance
2. The scheme is verified by comparing results with steady states from the EISMINT benchmark.

Future Work

1. Extend the model with Shallow Shelf Approximation (SSA) for simulating fast-flowing outlet glaciers and ice shelf system also known as "SIA+SSA hybrid" model
2. Compute optimal surface evolution using observation data and data assimilation method and re-define level set function $\varphi(\boldsymbol{x},t)$ for prediction

References

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