2009 03 16 Math 124 Midterm #2 KEY

I [9] Place the answers in the blanks to the right of the question:

1. In sequences described by formulae like $a_n = \dfrac{n}{2n-3}$ the domain of a_n is $\mathbb{N}$ or $\mathbb{W}$

2. What do the terms of the sequence in #1 converge to? $\lim_{n\to\infty} \frac{n}{2n-3} = \lim_{n\to\infty} \frac{1}{2-\frac{3}{n}} = \frac{1}{2}$

3. Why doesn't the series $\sum_{i=1}^{\infty} (^-2)^i$ converge? $\lim_{i\to\infty} (-2)^i$ does not exist. OR Geometric $|r| > 1$

4. $\sum_{n=1}^{\infty} (^-1)^n (4 - \frac{2}{n})$ is an alternating series of decreasing terms, yet it does not converge. Why not?
because $\lim_{n\to\infty} 4 - \frac{2}{n} = 4 \neq 0$

5. What is the difference between a Taylor series and a Maclaurin series? The centre of a Maclaurin series is 0

6. $e^{i\theta} =$ $\cos\theta + i\sin\theta$ 7. $\sqrt{4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})} =$ $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$

8. Simplify $\frac{1}{i}$ $\frac{i}{i} = \frac{i}{-1} = -i$ 9. $i^{4n} =$ 1

II [6] Find the limits of the sequences (or say they do not exist) SHOW WORK

10. $\left\{\dfrac{\ln n}{n}\right\}$

$\lim_{n\to\infty} \frac{\ln n}{n}$ $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

11. $\left\{\dfrac{\sin n\pi}{n}\right\}$

$= 0, 0, 0, 0$

$\to 0$

12. $a_n = \sqrt{\dfrac{n+1}{2n}}$

$\lim_{n\to\infty} \sqrt{\frac{n+1}{2n}} = \lim_{n\to\infty} \sqrt{\frac{1+\frac{1}{n}}{2}}$

$= \sqrt{\frac{1}{2}}$

$= \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

III [9] Test for convergence. SHOW WORK

13. $\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$

$u = \ln x$
$du = \frac{1}{x}dx$ Use $\int \frac{1}{u} du$

Consider $\int_{n=2}^{\infty} \frac{1}{x\ln x} = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x} dx$

$= \lim_{t\to\infty} \ln|\ln x| \Big]_2^t$

$= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2|$

$= \infty$

Diverges

14. $\sum_{n=1}^{\infty} \dfrac{n^2}{e^n}$ Ratio $\left|\frac{t_{n+1}}{t_n}\right|$

Consider $\lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{e}$

$= \frac{1}{e} < 1$

so it converges

15. $\sum_{k=1}^{\infty} \dfrac{k!}{k^k}$ Again a ratio

$\lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!}$

$= \lim_{k\to\infty} \frac{(k+1)k!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!}$

$= \lim_{k\to\infty} \frac{k^k}{(k+1)^k}$

$= \lim_{k\to\infty} \frac{1}{\left(\frac{k+1}{k}\right)^k}$

$= \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k}$

$= \frac{1}{e} < 1$ converges.

IV [6] Determine whether the following converge absolutely, conditionally, or not at all

[4] 16. $\sum_{n=1}^{\infty} (^-1)^n \dfrac{1}{2n+1}$

This alternating sequence of terms does $\to 0$ so it converges as is. However, if we do a limit comparison test absolutely with $\frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \frac{1}{2} > 0$ so, like the harmonic $\sum \frac{1}{2n+1}$ diverges.

Conclusion Conditional convergence

[2] 17. $\sum_{k=1}^{\infty} (^-1)^k \dfrac{2^k}{k}$

At last, a really easy one

Since $\frac{2^k}{k} \to \infty$ this one diverges

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V [8]What are the interval and radius of convergence?

19. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{(x-10)^n}{5^n}\right)$

This is an alternating series, so converges if $\left[\frac{(x-10)}{5}\right]^n$ is decreasing, ie if

$\left|\frac{x-10}{5}\right|<1$

$|x-10|<5$ $\quad R=5$

$x-10<5 \quad -x+10<5$
$x<15 \quad -x<-5$
$x>5$

if $x=15$ we get $(-1)^n(1)^n=(-1)^n$ Diverges

if $x=5$ we get $(-1)^n(-1)^n=1^n$ Diverges

$\therefore I = (5, 15)$

20. $\sum_{n=1}^{\infty}(-1)^n\frac{n^{2n}}{(1+3n^2)^n}$

Test with a root test

$\lim_{n\to\infty}\sqrt[n]{\frac{n^{2n}}{(1+3n^2)^n}} = \lim_{n\to\infty}\frac{n^2}{1+3n^2}\frac{\times\frac{1}{n^2}}{\times\frac{1}{n^2}}$

$=\lim_{n\to\infty}\frac{1}{\frac{1}{n^2}+3}=\frac{1}{3}<1$

So this series converges absolutely, and since $\nexists x$, it does so $\forall x$.

VI [4] Do ONE of these, clearly indicating which one you want marked.

21. Actually construct the Maclaurin series for sin x.

22. Show by differentiating the series for e^x that e^x is its own derivative.

23. Find the cube roots of $-i$.

(21) $f(x)=\sin x \quad f(0)=0$
$f'(x)=\cos x \quad f'(0)=1$
$f''(x)=-\sin x \quad f''(0)=0$
$f'''(x)=-\cos x \quad f'''(0)=-1$

So $\sum\frac{f^{(n)}(x)}{n!}x^n$

$=\frac{0}{0!}x^0+\frac{1}{1!}x^1+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\dots$

$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$

(22) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

$\frac{d}{dx}=0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\dots$

$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

$=e^x$

(23) $-i^{\frac{1}{3}}=(0+-i)^{\frac{1}{3}}=\left[1\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)\right]^{\frac{1}{3}}$

(a) $1^{\frac{1}{3}}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=0-i=-i$

(b) $1^{\frac{1}{3}}\cos\frac{1}{3}\left(\frac{7\pi}{2}\right)+i\sin\frac{1}{3}\left(\frac{7\pi}{2}\right)$
$=\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$

(c) $1^{\frac{1}{3}}\cos\frac{1}{3}\left(\frac{11\pi}{2}\right)+i\sin\frac{1}{3}\left(\frac{11\pi}{2}\right)$
$=\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

VII [8] Do TWO of these, clearly indicating which ones you want marked.

24. Use the Taylor inequality to estimate the accuracy of the Taylor approximation T_3 for sin x near $a=\frac{\pi}{6}$ on the interval $0\le x\le\frac{\pi}{3}$. $R_n(x)\le\frac{M}{(n+1)!}(x-a)^{n+1}\le\frac{1}{4!}\left(\frac{\pi}{6}\right)^4=.003$

[all derivatives are sin, cos, so $|\ |\le 1$ and $\max\left(x-\frac{\pi}{6}\right)=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$

25. Write out the first three terms of the Taylor series for $f(x)=\sqrt{x}$ for a = 3.

26. Show by doing the conversion to polar form how to find $(1-i)^{-10}$

27. A ball propelled up from the ground to height of 90m and on each bounce it reaches two-thirds its previous height. What is the total vertical distance it travels?

28. Find the actual sum of the convergent series $\sum_{n=1}^{\infty}\left(\frac{2^{n+1}}{5^n}\right)$

29. Estimate $(1.05)^{\frac{1}{2}}$ correct to .001 by a binomial expansion.

30. Find a power series expansion for $\frac{3}{2+x}$

(25) $f(x)=x^{\frac{1}{2}} \quad f(3)=\sqrt{3}$
$f'(x)=\frac{1}{2}x^{-\frac{1}{2}} \quad f'(3)=\frac{1}{2\sqrt{3}}$
$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}} \quad f''(3)=\frac{-1}{12\sqrt{3}}$
$f'''(x)=\frac{3}{8}x^{-\frac{5}{2}} \quad f'''(3)=\frac{-1}{72\sqrt{3}}$

so $T_3=\sqrt{3}+\frac{1}{2\sqrt{3}}\frac{(x-3)}{1!}-\frac{1}{12\sqrt{3}}\frac{(x-3)^2}{2!}$

(27) This time $a_1=180$ $r=\frac{2}{3}$ and $S=\frac{a_1}{1-r}=\frac{180}{1-\frac{2}{3}}=540m$

(26) $(1-i)^{10}=\left[\sqrt{2}\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right]^{10}$

$=\frac{1}{32}\left(\cos\frac{70\pi}{4}+i\sin\frac{70\pi}{4}\right)$

$=\frac{1}{32}\left(\cos\frac{-3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

$=\frac{1}{32}(0+i)$

$=\frac{1}{32}i$

(28) Factor out a 2 and

$S=2\sum\left(\frac{2}{5}\right)^n$ whence we have a geometric series with $a_1=\frac{2}{5}$ and $r=\frac{2}{5}$ so $S=2\left(\frac{a_1}{1-r}\right)$

$2\left(\frac{\frac{2}{5}}{1-\frac{2}{5}}\right)=2\left(\frac{\frac{2}{5}}{\frac{3}{5}}\right)=2\left(\frac{2}{3}\right)=\frac{4}{3}$

(29) $(1.05)^{\frac{1}{2}}=(1+x)^{\frac{1}{2}}=\frac{1}{0!}x^0+\frac{\frac{1}{2}x^1}{1!}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^2}{2!}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{3!}$ $\quad x=\frac{1}{20}$

$=1+\frac{1}{2}\left(\frac{1}{20}\right)-\frac{1}{8}\left(\frac{1}{400}\right)+\frac{3}{48}\left(\frac{1}{8000}\right)\dots$

$=1+.025-.0003125+.000007813$ not needed

$=1.025$

(30) $\frac{3}{2+x}=\frac{3}{2\left(1-\frac{-x}{2}\right)}=\frac{\frac{3}{2}}{1-\frac{-x}{2}}$ Geometric

Now this is $\sum_{n=1}^{\infty}\frac{3}{2}\left(\frac{-x}{2}\right)^{n-1}$ or $\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n$

(a) $e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\dots=1+i\theta-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}\dots$

$e^{-i\theta}=1-i\theta+\frac{(-i\theta)^2}{2!}+\frac{(-i\theta)^3}{3!}+\dots=1-i\theta-\frac{\theta^2}{2!}+\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}\dots$

subtract and divide by 2 yields $i\theta-\frac{i\theta^3}{3!}+\frac{i\theta^5}{5!}-\dots=i\sin\theta$

(b) $e^{i\theta}=\cos\theta+i\sin\theta$
$e^{-i\theta}=\cos\theta+i\sin(-\theta)$ } Subtracting and dividing by 2 yields $i\sin\theta$

VIII [+3] Bonus (not marked unless you have 75% above)

31. Simplify (a) using series $\frac{e^{i\theta}-e^{-i\theta}}{2}$ (b) without series